

## On quantum deformations of the simplest Lie parasuperalgebra

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1993 J. Phys. A: Math. Gen. 26 4311

(<http://iopscience.iop.org/0305-4470/26/17/036>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.68

The article was downloaded on 01/06/2010 at 19:32

Please note that [terms and conditions apply](#).

## On quantum deformations of the simplest Lie parasuperalgebra

J Beckers and N Debergh†

Theoretical and Mathematical Physics, Institute of Physics (B.5), University of Liège, B-4000 Liège 1, Belgium

Received 22 December 1992, in final form 17 March 1993

**Abstract.** Quantum deformations of the simplest Lie parasuperalgebra  $\text{Psqm}(2)$  are constructed by taking advantage of the well known representations of  $\text{su}_q(2, \mathbb{C})$ . As a particular result, we show that the fundamental Witten superalgebra  $\text{sqm}(2)$  cannot be deformed.

### 1. Introduction

The simplest ‘quantum group’  $\text{su}_q(2, \mathbb{C})$ , i.e. the  $q$ -deformation of the Lie algebra corresponding to  $\text{SU}(2, \mathbb{C})$ , has been extensively studied as an important example in connection with different fields in theoretical and mathematical physics [1–4]. More specifically, Macfarlane [3] has exploited the Sklyanin quantum algebras [5] in order to generalize the quantum theory of angular momentum and, correlatively, in order to develop a  $q$ -deformation of the quantum harmonic oscillator formalism.

The general method which has to be applied to any semisimple Lie algebra or superalgebra for constructing its quantum deformation is already known [2]. It does not work in the non-semisimple contexts although some results have been obtained through contraction procedures like those discussed for study Poincaré deformations [6].

Physically interesting Lie parasuperalgebras [7] appear as such non-semisimple structures when their *quadratic* character is enhanced by searching their connection with Sklyanin’s developments [5]. In fact, this happens in parasupersymmetric quantum mechanics (PSSQM) [8–10] subtended by the simplest Lie parasuperalgebra  $\text{Psqm}(2)$  [7].

We propose hereafter to discuss the possible  $q$ -deformations of  $\text{Psqm}(2)$  in connection with *arbitrary* orders  $p$  of paraquantization [11, 12] by exploiting the original  $\text{su}_q(2, \mathbb{C})$ -representations [4]. With new  $p$ -dependent general expressions of the parasupercharges, we are led to deformed relations characterizing these quantum deformations. In Section 2, we relate Lie parasuperalgebras and Sklyanin quadratic algebras and we notice that the simplest Lie parasuperalgebra  $\text{Psqm}(2)$  is nothing else but a direct sum of  $\text{su}(2, \mathbb{C})$  with an abelian (one-dimensional) algebra due to the conserved character of the parasupercharges. In Section 3, we construct the two new parasupercharges and study the possible deformed structure relations which can be

† Chercheur, Institut Interuniversitaire des Sciences Nucléaires, Bruxelles.

obtained for arbitrary orders  $p$  of paraquantization by dealing with the matrix realization of the deformed representations of  $su_q(2, \mathbb{C})$ . Section 4 is devoted to some comments related to the lowest orders  $p = 1, 2, 3$  and to recent results on  $(p + 1)$ -level systems studied by Semenov and Chumakov [15].

**2. Parasuperalgebras, quadratic algebras and the specific context of Psqm(2)**

Let us start by dealing with parasupersymmetric developments [8, 9] superposing bosons and  $p = 2$ -parafermions where  $p$  is the order of paraquantization. More precisely, let us consider the context of *exact* parasupersymmetry [9, 10] as a direct generalization of the Witten supersymmetric quantum mechanics [13] when two odd supercharges are involved (as it is necessary for oscillatorlike interactions in particular). The corresponding Lie parasuperalgebras [7] are generated by *even* operators denoted here by  $X_k (k = 1, \dots, n)$  and *odd* operators denoted here by  $Y_\alpha (\alpha = 1, \dots, m)$ , so that the corresponding structure relations are summarized by

$$[X_j, X_k] = c_{jk}^m X_m \quad [X_j, Y_\alpha] = d_{ja}^\beta Y_\beta \tag{2.1}$$

$$[Y_\alpha, [Y_\beta, Y_\gamma]] = e_{\alpha\beta}^j \{X_j, Y_\gamma\} - e_{\alpha\gamma}^j \{X_j, Y_\beta\}. \tag{2.2}$$

The Lie bracket of these parasuperalgebras appears as a double commutator leading to *trilinear* products only well defined in enveloping algebras or in ternary algebras.

Let us point out here that these structures reduce to quadratic algebras [5] by defining the commutators of two odd operators as new even generators denoted hereafter by  $Z$ :

$$[Y_\beta, Y_\gamma] = Z_{\beta\gamma}. \tag{2.3}$$

It is then easy to convince ourselves that Lie parasuperalgebras take *quadratic* forms. In fact, by using the corresponding Jacobi identity, we get the following commutation relations besides eqs. (2.1) and (2.3) for the above structures:

$$[Y_\alpha, Z_{\beta\gamma}] = e_{\alpha\beta}^j \{X_j, Y_\gamma\} - e_{\alpha\gamma}^j \{X_j, Y_\beta\} \tag{2.2'}$$

$$[X_j, Z_{\beta\gamma}] = d_{j\beta}^\alpha Z_{\beta\alpha} - d_{j\gamma}^\alpha Z_{\gamma\alpha} \tag{2.4}$$

$$[Z_{\alpha\beta}, Z_{\gamma\delta}] = (e_{\gamma\beta}^m d_{m\delta}^\epsilon - e_{\delta\beta}^m d_{m\gamma}^\epsilon) \{Y_\alpha, Y_\epsilon\} + e_{\delta\beta}^m \{X_m, Z_{\gamma\alpha}\} + (e_{\delta\alpha}^j d_{j\gamma}^\epsilon - e_{\gamma\alpha}^j d_{j\delta}^\epsilon) \{Y_\beta, Y_\epsilon\} - e_{\delta\alpha}^j \{X_j, Z_{\gamma\beta}\} + e_{\gamma\alpha}^j \{X_j, Z_{\delta\beta}\} - e_{\gamma\beta}^m \{X_m, Z_{\delta\alpha}\}. \tag{2.5}$$

These relations taken together with eqs. (2.1) and (2.3) appear in general as commutation relations of a quadratic algebra.

Let us now take the case of the simplest Lie parasuperalgebra called Psqm(2) [7]. It corresponds to a generalization of the Witten superalgebra sqm(2) [13]. This parasuperalgebra is generated by the even (parasuper) Hamiltonian  $H_{PSS}$  and the two odd (parasuper)charges  $Q$  and  $Q^\dagger$ . We have

$$[H_{PSS}, Q] = 0 \quad [H_{PSS}, Q^\dagger] = 0 \tag{2.6}$$

$$[Q, [Q^\dagger, Q]] = 2QH_{PSS} \quad [Q^\dagger, [Q, Q^\dagger]] = 2Q^\dagger H_{PSS}.$$

By defining two even operators and two odd ones respectively by

$$X_1 \equiv H_{PSS} \quad Z_{21} \equiv [Q^\dagger, Q] \equiv 2J_3 \quad Y_1 \equiv Q \equiv J_- \quad Y_2 \equiv Q^\dagger \equiv J_+ \tag{2.7}$$

we immediately get the algebra

$$[X_1, J_\pm] = 0 \quad [X_1, J_3] = 0 \tag{2.8a}$$

$$[J_+, J_-] = 2J_3 \quad [J_3, J_\pm] = \pm J_\pm X_1. \tag{2.8b}$$

In fact, this is not a quadratic algebra (as expected) but the relations (2.8) refer to the direct sum  $su(2, \mathbb{C}) \oplus X_1$ , nevertheless characterized by the non-commutative co-product:

$$\begin{aligned} \Delta(J_+) &= J_+ \otimes X_1 + I \otimes J_+ & \Delta(J_-) &= J_- \otimes I + X_1 \otimes J_- \\ \Delta(J_3) &= J_3 \otimes X_1 + X_1 \otimes J_3 & \Delta(X_1) &= X_1 \otimes X_1. \end{aligned} \tag{2.9}$$

Such a property does not imply that the above structure is a quantum algebra: we have recently learned [14] that it is possible to get deformed coproducts without deforming the algebra itself.

In conclusion, the simplest parasuperalgebra  $\text{Psqm}(2)$  cannot be deformed through already known methods. We propose to exploit its connection with  $su(2, \mathbb{C})$  (through the above direct sum structure) in order to discuss its possible deformations for arbitrary orders of paraquantization.

### 3. Deformed $\text{Psqm}_q(2)$ from representations of $su_q(2, \mathbb{C})$

Due to the main role played by the parafermionic ‘variables’ in the construction of  $N=2$ -PSSQM, we propose now to study the possible deformations of our fundamental parasuperalgebra  $\text{Psqm}(2)$ . We want to exploit the explicit representations of  $su_q(2, \mathbb{C})$  inside the construction of the two (new) parasupercharges needed in the structure relations (2.6). In fact, after Macfarlane [3] and Biedenharn [4], let us recall that, within a basis of  $q$ -quantum angular momentum states  $|j, m\rangle_q$ , we have for each ( $j=p/2$ )-fixed value

$$J_{\pm} |j, m\rangle_q = ([j \mp m][j \pm m + 1])^{1/2} |j, m \pm 1\rangle_q \tag{3.1a}$$

$$J_3 |j, m\rangle_q = m |j, m\rangle_q \tag{3.1b}$$

where we choose [3]

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3.2}$$

By referring to the corresponding matrix realizations of the irreducible representations  $D_q^{(p/2)}$ , we can evidently point out the  $(p+1)$ -dimensional diagonal matrix  $(J_3) = m\mathbb{1}$  and the two non-diagonal ones associated with the scaling operators  $J_{\pm}$ , i.e. respectively

$$J_+ \rightarrow \sum_{j=1}^p ([j][p-j+1])^{1/2} e_{j,j+1} \tag{3.3a}$$

and

$$J_- \rightarrow \sum_{j=1}^p ([j][p-j+1])^{1/2} e_{j+1,j}. \tag{3.3b}$$

As usual our notations  $e_{j,k}$  correspond to  $(p+1)$ -dimensional matrices containing zeros everywhere except units at the intersection of the  $j$ th row and the  $k$ th column. Let us notice that the  $(p=1)$ -value (corresponding to the current fermionic context)

leads through eqs. (3.3) to the meaningful Pauli matrices  $\sigma_+$  and  $\sigma_-$  which are the fermionic 'variables' introduced in the Witten supercharges [13].

By taking care of these observations, we now propose general forms of the parasupercharges as follows

$$Q = \sum_{j=1}^p (\frac{1}{2}[j][p-j+1])^{1/2} (p_x + iW_j(x)) e_{j+1,j} \quad (3.4)$$

$$Q^\dagger = \sum_{j=1}^p (\frac{1}{2}[j][p-j+1])^{1/2} (p_x - iW_j(x)) e_{j,j+1}$$

where evidently  $p_x = -i(d/dx)$  and the  $W_j(x)$ 's are the implied (parasuper) potentials. Such parasupercharges include in their explicit forms the effect of the  $q$ -deformations issued from the  $su(2, \mathbb{C})$  developments. Due to their structures, they evidently satisfy (expected) nilpotent relations such that

$$Q^{p+1} = 0, (Q^\dagger)^{p+1} = 0 \quad (3.5)$$

and we can ask for deformed relations corresponding to eqs. (2.6) as well as for possible associated Hamiltonians where the latter have to be on the diagonal form with elements

$$H_j^{(q)} = \frac{1}{2} p^2 + f_j(x).$$

Due to the structure relations (2.6) with respect to double commutators, let us search for information on the real functions  $\alpha(q)$ ,  $\beta(q)$ ,  $\gamma(q)$  and  $\delta(q)$  as well as on the hermitian deformed Hamiltonians  $H_q$  inside the relations

$$\alpha(q) Q Q^\dagger Q - \beta(q) Q^2 Q^\dagger - \gamma(q) Q^\dagger Q^2 = \delta(q) Q H^{(q)} \quad (3.6a)$$

and

$$\alpha(q) Q^\dagger Q Q^\dagger - \beta(q) Q Q^\dagger Q - \gamma(q) Q^\dagger Q^2 = \delta(q) Q^\dagger H^{(q)} \quad (3.6b)$$

when the parasupercharges (3.4) are evidently introduced in these developments. Such structure relations immediately imply (as in the nondeformed context) typical constraints on the parasuperpotentials: these appear as  $(p-1)$ -Riccati equations given by

$$W_{j+1}^2 + W'_{j+1} = W_j^2 - W'_j + c_j \quad j = 1, 2, \dots, p-1 \quad (3.7)$$

where primes refer to space derivatives as usual and the  $c$ 's are arbitrary constants. Moreover we get  $(2p-2)$ -constraints on the three functions  $\alpha(q)$ ,  $\beta(q)$  and  $\gamma(q)$  which take the forms

$$\alpha(q)[p-2j+2] + \beta(q)[2j-p-4] + \gamma(q)[2j-p] = 0 \quad j = 2, 3, \dots, p \quad (3.8a)$$

and

$$\begin{aligned} \alpha(q)[j][p-j+1]c_{j-1} - \beta(q)[j-2][p-j+3]c_{j-2} \\ + \gamma(q)([2j-p]c_{j-1} - [j+1][p-j]c_j) = 0 \quad j = 2, 3, \dots, p. \end{aligned} \quad (3.8b)$$

Such results are easily obtained by using in particular the following identity:

$$[n-m+1] = [m][n] - [m-1][n+1]. \quad (3.9)$$

The discussion of the systems (3.8) is intimately related to the cancellation or not of the bracket  $[j]$  for specific values of  $j$  (such that  $j=2, 3$ ) or for arbitrary values of  $j=2, \dots, p+1$ . Let us point out, in particular, the specific interest of eq. (3.9) in these results. For example, if  $[2]=0$ , it implies  $[2l]=0$  for  $l=2, 3, \dots$ , so that we have only to consider odd  $p$ 's in the first class hereafter. In fact, we distinguish four classes of results, i.e.

(i)  $[2]=0, p$  odd,  $\alpha = -(\beta + \gamma)$

$$c_1 = -c_2, c_3 = -c_4, \dots, c_{p-2} = -c_{p-1}; \tag{3.10}$$

(ii)  $[2] \neq 0, [3]=0, \alpha = -\beta = -\gamma, c_j$  arbitrary if  $p \leq 3$

$$\text{or } c_j + c_{j+1} + c_{j+2} = 0 \text{ if } p > 3 \text{ (except if } p = 3n + j - 1, n \in N); \tag{3.11}$$

(iii)  $[2] \neq 0, [3] \neq 0, [j]=0, j=4, \dots, p+1$ , when  $p \geq 3$

$$\alpha = (q^2 + q^{-2})\beta \quad \beta = \gamma \tag{3.12}$$

the  $c_j$ 's being constrained but at least one of them being non-zero;

(iv)  $[j] \neq 0 \quad j=2, \dots, p+1$

$$\alpha = (q^2 + q^{-2})\beta \quad \beta = \gamma, c_1 = \dots = c_{p-1} = 0. \tag{3.13}$$

We mention that the case  $p=2$  is a particular one. Indeed such a context leads to the classes (ii) and (iv) where, in the last case,  $\alpha$  is left arbitrary. For a uniform treatment, we constrain it following eq. (3.13).

Let us now draw the conclusions on the possible  $q$ -deformed  $\text{Psqm}(2)$  which can be obtained from the above discussion in correspondence with these four classes. Besides the structure relations, we also give the explicit forms of the corresponding Hamiltonians which are characterized by the following (diagonal) elements:

$$\begin{aligned} H_{j,j}^{(q)} = & \frac{\alpha [j][p-j+1]}{\delta} \frac{1}{2} (p_x^2 + W_j^2 + W_j') \\ & - \frac{\beta [j-1][p-j+2]}{\delta} \frac{1}{2} (p_x^2 + W_{j-1}^2 - W_{j-1}') \\ & - \frac{\gamma [j+1][p-j]}{\delta} \frac{1}{2} (p_x^2 + W_j^2 + W_j' + c_j), j=1, \dots, p \end{aligned} \tag{3.14a}$$

and

$$H_{p+1,p+1}^{(q)} = \frac{\alpha [p]}{\delta} \frac{1}{2} (p_x^2 + W_p^2 - W_p') - \frac{\beta [2][p-1]}{\delta} \frac{1}{2} (p_x^2 + W_p^2 - W_p' - c_{p-1}). \tag{3.14b}$$

By using the definitions (3.4), the deformed relations (3.6), the constraints (3.7) and

the matrix elements (3.14), we get in correspondence with the four above classes that

$$\begin{aligned}
 \text{(i')} \quad Q^2 = 0, [\mathcal{H}, Q] = 0, QQ^\dagger Q = [p]Q\mathcal{H} \\
 \mathcal{H}_{1,1} = \frac{1}{2}(p_x^2 + W_1^2 + W_1') \\
 \mathcal{H}_{2,2} = \frac{1}{2}(p_x^2 + W_1^2 - W_1') = \frac{1}{2}(p_x^2 + W_2^2 + W_2' - c_1) \\
 \dots;
 \end{aligned} \tag{3.15}$$

$$\begin{aligned}
 \text{(ii')} \quad \text{if } p = 1, 3, 4: Q^2 = 0, [\mathcal{H}, Q] = 0, QQ^\dagger Q = Q\mathcal{H} \\
 \mathcal{H}_{1,1} = \frac{1}{2}(p_x^2 + W_1^2 + W_1' + [2][p-1]c_1) \\
 \dots;
 \end{aligned} \tag{3.16a}$$

if  $p \neq 1, 3, 4$ :

$$\begin{aligned}
 Q^{p+1} = 0, [\mathcal{H}, Q] = 0, Q^2Q^\dagger + Q^\dagger Q^2 + QQ^\dagger Q = Q\mathcal{H} \\
 \mathcal{H}_{1,1} = \frac{1}{2} \left( p_x^2 + W_1^2 + W_1' + \frac{[2][p-1]c_1}{[p] + [2][p-1]} \right) \\
 \dots;
 \end{aligned} \tag{3.16b}$$

$$\begin{aligned}
 \text{(iii')} \quad \text{if } j = 4: Q^{p+1} = 0, [\mathcal{H}, Q] = 0, \{Q^2, Q^\dagger\} = -[2][p-1]Q\mathcal{H} \\
 \mathcal{H}_{1,1} = \frac{1}{2}(p_x^2 + W_1^2 + W_1' + c_1) \\
 \dots;
 \end{aligned} \tag{3.17a}$$

if  $j \neq 4: Q^{p+1} = 0, [\mathcal{H}, Q] = 0$

$$\begin{aligned}
 [Q, [Q^\dagger, Q]_q]_q = ([p+2] - [p])Q\mathcal{H} \quad \text{with } [A, B]_q \equiv qAB - \frac{1}{q}BA \\
 \mathcal{H}_{1,1} = \frac{1}{2} (p_x^2 + W_1^2 + W_1') - \frac{1}{2} \frac{[2][p-1]c_1}{[p+2] - [p]} \\
 \dots;
 \end{aligned} \tag{3.17b}$$

$$\begin{aligned}
 \text{(iv')} \quad Q^{p+1} = 0, [H_{PSS}, Q] = 0 \\
 [Q, [Q^\dagger, Q]_q]_q = ([p+2] - [p])QH_{PSS}.
 \end{aligned} \tag{3.18}$$

By noticing that hermitian conjugate structure relations have to be included and that two successive diagonal matrix elements of the Hamiltonian(s) have to be superpartners related through the constraints (3.7), we have limited the above results to a minimal listing.

#### 4. Some comments and conclusions

Let us illustrate the contents of the above results (3.15)–(3.18) by considering the first orders  $p = 1, 2, 3$  of paraquantization in connection with recent studies.

If  $p = 1$ , the classes (i'), (ii') and (iv') are concerned with and they lead to the same structure characterized by

$$Q^2 = 0 \quad [H_{SS}, Q] = 0 \quad QQ^\dagger Q = QH_{SS} = Q\{Q^\dagger, Q\} \quad (4.1)$$

where evidently (through the superposition of bosons and fermions) we are dealing with supersymmetric considerations and a supersymmetric Hamiltonian  $H_{SS}$  [13]. From (3.18), we learn that the deformation of  $\text{sqm}(2)$  is not feasible for any value of  $q$  in complete agreement with a recent result [16] showing that the Spiridonov approach of such a problem [17] is nothing else than a standard Witten supersymmetric model but with a  $q$ -dependence inside the superpotentials.

If  $p = 2$ , only the classes (ii') and (iv') are implied in correspondence with  $[3] = 0$  and  $[3] \neq 0$  respectively. We immediately get from (3.16b) that

$$Q^3 = 0 \quad [H, Q] = 0 \quad QQ^\dagger Q + Q^2 Q^\dagger + Q^\dagger Q^2 = 2[2]QH \quad (4.2)$$

where

$$H = \text{diag}(H_{1,1}, H_{2,2}, H_{3,3}) \quad (4.3a)$$

$$H_{1,1} = \frac{1}{2}(p_x^2 + W_1^2 + W_1' + \frac{1}{2}c_1) \quad (4.3b)$$

$$H_{2,2} = \frac{1}{2}(p_x^2 + W_1^2 - W_1' - \frac{1}{2}c_1) = \frac{1}{2}(p_x^2 + W_2^2 + W_2' - \frac{1}{2}c_1) \quad (4.3c)$$

$$H_{3,3} = \frac{1}{2}(p_x^2 + W_2^2 - W_2' - \frac{1}{2}c_1). \quad (4.3d)$$

This parasupersymmetric Hamiltonian coincides with the original one constructed by Rubakov and Spiridonov [8] and the above structure is a  $q$ -deformation of their parasuperalgebra. Let us point out that  $H_{1,1}$  with  $H_{2,2}$  and  $H_{2,2}$  with  $H_{3,3}$  appear as (expected) respective superpartners on the basis of the constraints (3.7). When  $[3] \neq 0$ , we immediately get, from (3.18), that

$$[Q, [Q^\dagger, Q]_q]_q = ([4] - [2])QH_{PSS}$$

i.e.

$$[Q, [Q^\dagger, Q]] = [2]QH_{PSS}. \quad (4.4)$$

This context is exactly the one discussed elsewhere [9, 10] characterized by the cancellation of the arbitrary constant  $c_1$ , leading in particular to the direct exploitation of Riccati equations in [18].

The above two cases corresponding respectively to  $[3] = 0$  and  $[3] \neq 0$  just fall into the Semenov–Chumakov discussion [15] of the so-called  $N (= p + 1)$ -level systems. When  $[3] = 0$ , the Rubakov–Spiridonov model [8] corresponds to the interaction between a bosonic mode and a 3-level system (of the  $\Xi$ -type) while, when  $[3] \neq 0$ , we recover our model of the  $\Lambda$ - or  $\vee$ -type with  $W_1 = -W_2 = x$ ,  $c_1 = 0$ , for the harmonic oscillatorlike context [9, 10].

If  $p = 3$ , the four classes have to give  $q$ -deformations of  $\text{Psqm}(2)$ . These results lead to examples of 4-level systems of  $a$ -type (when  $[4] = 0$ ,  $c_1 = c_2 = 2$ ),  $b$ -type (when  $c_1 = c_2 = 0$ ) and  $e$ -type (when  $[3] = 0$ ,  $c_1 = 2$ ,  $c_2 = 0$ ) following the Semenov–Chumakov notations.

### Acknowledgments

Thanks are due to the referee for pertinent remarks.



## References

- [1] Drinfeld V G 1986 *Proc. ICM Berkeley, Ca* ed A M Gleason (Providence, RI; AMS) p. 798  
Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 **11** 247; 1987 *Comm. Math. Phys.* **102** 537
- [2] Woronowicz S 1987 *Publ. RIMS (Kyoto University)* **23** 117; 1987 *Comm. Math. Phys.* **111** 613  
Pasquier V 1988 *Nucl. Phys. B* **295** 491; 1988 *Comm. Math. Phys.* **118** 355
- [3] Macfarlane A J 1989 *J. Phys. A* **22** 4581
- [4] Biedenharn L C 1989 *J. Phys. A* **22** L873
- [5] Sklyanin E K 1982 *Funkt. Anal. Pril.* **16** 27 [1983 **16** 263]  
Sklyanin E K 1983 *Funkt. Anal. Pril.* **17** 34 [1984 **17** 273]
- [6] Lukierski J, Ruegg H, Nowicki A and Tolstoy V 1991 *Phys. Lett. B* **264** 331  
Giller S, Kosinski P, Majewski M, Maslanka P and Kunz J 1992 *Phys. Lett. B* **286** 57
- [7] Beckers J and Debergh N 1990 *J. Phys. A* **23** L751S, L1073
- [8] Rubakov V A and Spiridonov V P 1988 *Mod. Phys. Lett. A* **3** 1337
- [9] Beckers J and Debergh N 1990 *Nucl. Phys. B* **340** 767
- [10] Beckers J and Debergh N 1991 *Lect. Notes in Physics* **382** 414
- [11] Wigner E P 1950 *Phys. Rev.* **77** 711  
Green H S 1953 *Phys. Rev.* **90** 270
- [12] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Tokyo: University of Tokyo Press)
- [13] Witten E 1981 *Nucl. Phys. B* **188** 513
- [14] Karimipour V 1992 *Remarks on Multiparametric Quantum (Super)Algebras* preprint SUTDP/92/70 (June)
- [15] Semenov V V and Chumakov S M 1991 *Phys. Lett. B* **262** 451
- [16] Ndimubandi J 1993 *Mod. Phys. Lett. A* **8** 429
- [17] Spiridonov V P 1992 *Mod. Phys. Lett. A* **7** 1241
- [18] Beckers J and Debergh N 1991 *Z. Phys. C: Particles and Fields* **51** 519